

COMPLEMENTS OF HYPERPLANE SUB-BUNDLES IN PROJECTIVE SPACE BUNDLES OVER \mathbb{P}^1

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ABSTRACT. We establish that the isomorphism type as an abstract algebraic variety of the complement of an ample hyperplane sub-bundle H of a \mathbb{P}^{r-1} -bundle $\mathbb{P}(E) \rightarrow \mathbb{P}^1$ depends only on the r -fold self-intersection $(H^r) \in \mathbb{Z}$ of H . In particular it depends neither on the ambient bundle $\mathbb{P}(E)$ nor on the choice of a particular ample sub-bundle with given r -fold self-intersection. Our proof exploits the unexpected property that every such complement comes equipped with the structure of a non trivial torsor under a vector bundle over the affine line with a double origin.

INTRODUCTION

The Danilov-Gizatullin Isomorphism Theorem [5, Theorem 5.8.1] (see also [2, 4] for short self-contained proofs) is a surprising result which asserts that the isomorphism type as an abstract algebraic variety of the complement of an ample section C of a \mathbb{P}^1 -bundle $\bar{\nu} : \mathbb{P}(E) \rightarrow \mathbb{P}^1_C$ over the complex projective line depends only on the self-intersection $(C^2) \geq 2$ of C . In particular, it depends neither on the ambient bundle nor on the chosen ample section with fixed self-intersection d . For such a section, the locally trivial fibration $\nu : \mathbb{P}(E) \setminus C \rightarrow \mathbb{P}^1$ induced by the restriction of the structure morphism $\bar{\nu}$ is homeomorphic in the euclidean topology to the complex line bundle $\mathcal{O}_{\mathbb{P}^1}(-d) \rightarrow \mathbb{P}^1$. However the non vanishing of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$ for $d \geq 2$ implies that $\nu : \mathbb{P}(E) \setminus C \rightarrow \mathbb{P}^1$ is in general a non trivial algebraic $\mathcal{O}_{\mathbb{P}^1}(-d)$ -torsor, and the ampleness of C is in fact precisely equivalent to its non triviality. So the Danilov-Gizatullin Theorem can be rephrased as the fact that the isomorphism type as an abstract algebraic variety of the total space of a non trivial $\mathcal{O}_{\mathbb{P}^1}(-d)$ -torsor is uniquely determined by its underlying structure of topological complex line bundle over \mathbb{P}^1 .

More generally, given a vector bundle $E \rightarrow \mathbb{P}^1$ of rank $r \geq 3$ and a sub-vector bundle $F \subset E$ of corank one with quotient line bundle L , the complement in the projective bundle $\bar{\nu} : \mathbb{P}(E) \rightarrow \mathbb{P}^1$ of lines in E of the hyperplane sub-bundle $H = \mathbb{P}(F)$ inherits the structure of an $F \otimes L^{-1}$ -torsor $\nu : \mathbb{P}(E) \setminus H \rightarrow \mathbb{P}^1$. Similarly as above, the latter is homeomorphic in the euclidean topology to the complex vector bundle $F \otimes L^{-1} \rightarrow \mathbb{P}^1$. Furthermore, $F \otimes L^{-1}$ is homeomorphic as a complex vector bundle to $\det(F \otimes L^{-1}) \oplus \mathbb{A}_{\mathbb{P}^1}^{r-2} \simeq \mathcal{O}_{\mathbb{P}^1}(-(H^r)) \oplus \mathbb{A}_{\mathbb{P}^1}^{r-2}$, where $(H^r) \in \mathbb{Z}$ denotes the r -fold intersection product of H with itself. So one may ask by analogy with the one dimensional case if for an ample H , the integer $(H^r) \geq r$ uniquely determines the isomorphism type of $\mathbb{P}(E) \setminus H$ as an algebraic variety. However, since the ampleness of H is in general no longer equivalent to the non triviality of the torsor $\nu : \mathbb{P}(E) \setminus H \rightarrow \mathbb{P}^1$, the following problem seems more natural:

Question. *Is the isomorphism type as an abstract algebraic variety of the total space of a nontrivial torsor under an algebraic vector bundle $G \rightarrow \mathbb{P}^1$ uniquely determined by the isomorphism type of G as a topological complex vector bundle over \mathbb{P}^1 , that is, by the rank and the degree of G ?*

While our results imply in particular that non trivial torsors under homeomorphic complex vector bundles do indeed have isomorphic total spaces, the answer to this question is negative in general: there exists torsors under non homeomorphic vector bundles or rank $r \geq 2$ which have isomorphic total spaces as algebraic varieties. For instance, the complement of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ is an affine surface S which inherits two structures of non trivial $\mathcal{O}_{\mathbb{P}^1}(-2)$ -torsor via the first and the second projections $\text{pr}_i : S \rightarrow \mathbb{P}^1$, $i = 1, 2$. The Picard group of S is isomorphic to \mathbb{Z} and for every $k \in \mathbb{Z}$ the line bundles $\text{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(k)$ and $\text{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(-k)$ are isomorphic. It follows that for every $k \in \mathbb{Z}$, the total space of $\text{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(k)$ is simultaneously the total space of an $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(k)$ -torsor and of an $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-k)$ -torsor over \mathbb{P}^1 via the first and the second projection respectively. In particular, for every $k \neq 0$, we obtain an affine variety which is simultaneously the total space of non trivial torsors under complex vector bundles over \mathbb{P}^1 with different topological types.

So the degree of the complex vector bundle $G \rightarrow \mathbb{P}^1$ is not the appropriate numerical invariant to classify isomorphism types of total spaces of non trivial algebraic G -torsors. In contrast, our main result can be summarized as follows:

Theorem. *The total space of a torsor $\nu : V \rightarrow \mathbb{P}^1$ under a vector bundle $G \rightarrow \mathbb{P}^1$ is an affine variety if and only if $\nu : V \rightarrow \mathbb{P}^1$ is a non trivial torsor. If so, the isomorphism type of V as an abstract algebraic variety is uniquely determined by the rank of G and the absolute value of $\deg G + 2$.*

The role of the integer $|\deg G + 2|$ may look surprising but the latter is intimately related to the subgroup of the Picard group $\text{Pic}(V) \simeq \mathbb{Z}$ of V generated by the canonical bundle $K_V = \det(\Omega_V^1)$ of V . Indeed, for a G -torsor $\nu : V \rightarrow \mathbb{P}^1$, the relative cotangent bundle $\Omega_{V/\mathbb{P}^1}^1$ is isomorphic to $\nu^* G^\vee$ and so it follows from the relative cotangent exact sequence

$$0 \rightarrow \nu^* \Omega_{\mathbb{P}^1}^1 \rightarrow \Omega_V^1 \rightarrow \Omega_{V/\mathbb{P}^1}^1 \simeq \nu^* G^\vee \rightarrow 0$$

that $K_V \simeq \nu^*(\det G^\vee \otimes \Omega_{\mathbb{P}^1}^1)$ whence that the subgroup of $\text{Pic}(V)$ generated by K_V is isomorphic to $|\deg G + 2| \mathbb{Z} \subset \mathbb{Z}$.

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In the particular case where $\nu : V \rightarrow \mathbb{P}^1$ is a torsor arising as the complement $\mathbb{P}(E) \setminus H$ of an ample hyperplane sub-bundle H , one has $\deg G = -(H^r) \leq -r$ and so the above characterization specializes to the following generalization of the geometric form of the Danilov-Gizatullin Theorem:

Corollary. *The isomorphism type as an abstract algebraic variety of the complement of an ample hyperplane sub-bundle H of a \mathbb{P}^{r-1} -bundle $\rho : \mathbb{P}(E) \rightarrow \mathbb{P}^1$ depends only on the r -fold self-intersection $(H^r) \geq r$ of H .*

The proof of the above results exploits a hidden and unexpected feature of total spaces of non trivial torsors $\nu : V \rightarrow \mathbb{P}^1$ under a vector bundle $G \rightarrow \mathbb{P}^1$ of rank $r \geq 1$, namely the existence on every such V of the structure $\rho : V \rightarrow X$ of a non trivial torsor under a vector bundle $\tilde{G} \rightarrow X$ of the same rank r , on a non separated scheme X , isomorphic to the affine line with a double origin. The structure of these bundles $\tilde{G} \rightarrow X$ is very similar to that of vector bundles on \mathbb{P}^1 : in particular the existence of a covering of X by two open subsets isomorphic to \mathbb{C} and intersecting along \mathbb{C}^* implies that as a topological complex vector bundle, $\tilde{G} \rightarrow X$ is uniquely determined by a homotopy class of maps $S^1 \rightarrow \mathrm{GL}_r(\mathbb{C})$, whence simply by the “degree” of its determinant. In this setting, we establish that the total space of a non trivial torsor $\nu : V \rightarrow \mathbb{P}^1$ under a vector bundle $G \rightarrow \mathbb{P}^1$ of degree d carries the structure of a non trivial torsor $\rho : V \rightarrow X$ under a vector bundle $\tilde{G} \rightarrow X$ of degree $d+2$ uniquely determined by G . While there exists infinite moduli for isomorphism type of total spaces of non trivial torsors under a line bundle on X , the Danilov-Gizatullin Theorem can be re-interpreted as the fact that the total spaces of non trivial torsors $\nu : V \rightarrow \mathbb{P}^1$ under $G = \mathcal{O}_{\mathbb{P}^1}(-d)$ are all isomorphic as torsors $\rho : V \rightarrow X$ under the corresponding line bundle \tilde{G} . For higher dimensional vector bundles $\tilde{G} \rightarrow X$, we establish in contrast that the isomorphism type as an abstract variety of the total space of a non trivial \tilde{G} -torsor $\rho : V \rightarrow X$ is uniquely determined by the absolute value of the degree of \tilde{G} .

The article is organized as follows: the first section recalls basic properties of torsors under vector bundles. Then section two is devoted to the study of isomorphism types of total spaces of non trivial torsors under vector bundles on the affine line with a double origin. These results are applied in the third section to the case of torsors under vector bundles on \mathbb{P}^1 .

1. RECOLLECTION ON AFFINE-LINEAR BUNDLES

Here, to fix the notation and convention that will be used in the sequel, we briefly recall classical facts about vector bundles, projective bundles and affine-linear bundles.

1.1. Vector bundles and projective bundles.

1.1.1. A *vector bundle of rank $r \geq 1$* on a scheme X is the relative spectrum $p : F = \mathrm{Spec}_X(\mathrm{Sym}(\mathcal{F}^\vee)) \rightarrow X$ of the symmetric algebra of the dual of a locally free coherent \mathcal{O}_X -module \mathcal{F} of rank r . The X -scheme F represents the contravariant functor from the category of schemes over X to the category of abelian groups which associates to an X -scheme $f : Y \rightarrow X$ the group $\Gamma(Y, f^*\mathcal{F})$ of global sections of $f^*\mathcal{F}$ over Y . It follows in particular that $p : F \rightarrow X$ is a locally constant commutative affine group scheme over X , with group law $F \times_X F \rightarrow F$ induced by the diagonal homomorphism of \mathcal{O}_X -module $\mathcal{F}^\vee \rightarrow \mathcal{F}^\vee \oplus \mathcal{F}^\vee$.

1.1.2. Given a vector bundle $p : E = \mathrm{Spec}_X(\mathrm{Sym}(\mathcal{E}^\vee)) \rightarrow X$, the *projective bundle of lines in E* is the relative $\mathrm{proj} \, \mathcal{P} : \mathbb{P}(E) = \mathrm{Proj}_X(\mathrm{Sym}(\mathcal{E}^\vee)) \rightarrow X$ of the symmetric algebra of \mathcal{E}^\vee considered as a graded quasi-coherent \mathcal{O}_X -algebra $\mathrm{Sym}(\mathcal{E}^\vee) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(\mathcal{E}^\vee)$. There is a canonical surjection $\mathcal{P}^*E^\vee \rightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ which yields dually a closed embedding $\mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow \mathcal{P}^*E$ of the tautological line bundle as a line sub-bundle of \mathcal{P}^*E . More generally, if $f : Y \rightarrow X$ is a scheme over X then an X -morphism $\tilde{f} : Y \rightarrow \mathbb{P}(E)$ is uniquely determined by a sub-line bundle of f^*E , which then coincides with $f^*\mathcal{O}_{\mathbb{P}(E)}(-1)$. In other word, $\mathbb{P}(E)$ represents the functor which associates to an X -scheme $f : Y \rightarrow X$ the set of sub-line bundles of f^*E . Recall that if $L = \mathrm{Spec}_X(\mathrm{Sym}(\mathcal{L}^\vee)) \rightarrow X$ is a line bundle on X , then the canonical isomorphism $\mathrm{Sym}((\mathcal{E} \otimes \mathcal{L})^\vee) \simeq \bigoplus_{n \geq 0} \mathrm{Sym}^n(\mathcal{E}^\vee) \otimes (\mathcal{L}^\vee)^{\otimes n}$ of graded \mathcal{O}_X -algebras yields an isomorphism $\varphi : \mathbb{P}(E) \xrightarrow{\sim} \mathbb{P}(E \otimes L)$ of schemes over X with $\varphi^*\mathcal{O}_{\mathbb{P}(E \otimes L)}(-1) \simeq \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \mathcal{P}^*L$.

1.2. Torsors under vector bundles.

1.2.1. Torsors under a vector bundle on a scheme are the analogues in a relative setting of affine spaces attached to a vector space. Namely, given a vector bundle $p : F = \mathrm{Spec}_X(\mathrm{Sym}(\mathcal{F}^\vee)) \rightarrow X$, a *principal homogeneous F -bundle*, or an *F -torsor*, is a scheme $\nu : V \rightarrow X$ equipped with an action $\mu : F \times_X V \rightarrow V$ of the group scheme F for which there exists a covering of X by open subsets $\{U_i\}_{i \in I}$ such that for every $i \in I$, $V|_{U_i} = \nu^{-1}(U_i)$ is equivariantly isomorphic to $F|_{U_i}$ acting on itself by translations. Given a collection of equivariant trivializations $\tau_i : V|_{U_i} \xrightarrow{\sim} F|_{U_i}$, $i \in I$, it follows that for every $i, j \in I$, $\tau_i \circ \tau_j^{-1}|_{U_i \cap U_j}$ is an equivariant automorphism of $F|_{U_i \cap U_j}$ whence is a translation determined by a section $g_{ij} \in \Gamma(U_i \cap U_j, F) = \Gamma(U_i \cap U_j, \mathcal{F})$ of $F|_{U_i \cap U_j}$. Clearly, $g_{ik}|_{U_i \cap U_j \cap U_k} = g_{ij}|_{U_i \cap U_j \cap U_k} + g_{jk}|_{U_i \cap U_j \cap U_k}$ for every $i, j, k \in I$, that is, $(g_{ij})_{i,j \in I}$ is a Čech 1-cocycle with value in the sheaf \mathcal{F} for the open covering $\{U_i\}_{i \in I}$. Changing the trivializations τ_i replaces $(g_{ij})_{i,j \in I}$ by a cohomologous cocycle and the cohomology class is unaltered if V is replaced by an isomorphic torsor. Thus $\nu : V \rightarrow X$ defines a class $c(V) \in \check{H}^1(\{U_i\}_{i \in I}, F) = \check{H}^1(\{U_i\}_{i \in I}, \mathcal{F})$ and a standard argument eventually shows that there is a one-to-one correspondence between isomorphism classes of F -torsors and elements of the cohomology group $\check{H}^1(X, F) \simeq H^1(X, F)$, with $0 \in H^1(X, F)$ corresponding to the trivial F -torsor $p : F \rightarrow X$ (see e.g. [7, 16.4.9]). This implies in particular that every F -torsor on an affine scheme X is isomorphic to the trivial one.

1.2.2. Recall that the relative cotangent bundle $\Omega_{F/X}^1$ of a vector bundle $p : F \rightarrow X$ is canonically isomorphic to p^*F^\vee [7, 16.5.15]. One checks using the above local description that this holds more generally for any F -torsor $\nu : V \rightarrow X$, providing a canonical short exact sequence $0 \rightarrow \nu^*\Omega_X^1 \rightarrow \Omega_V^1 \rightarrow \Omega_{V/X}^1 \simeq \nu^*F^\vee \rightarrow 0$ of vector bundles on V . If X

is normal then the natural homomorphism $\nu^* : \text{Pic}(X) \xrightarrow{\sim} \text{Pic}(V)$ is an isomorphism; the relative canonical bundle $K_{V/X} = \det(\Omega_{V/X}^1)$ and the canonical bundle $K_V = \det(\Omega_V^1)$ of V then coincide respectively with the images by ν^* of the line bundles $\det(F^\vee)$ and $K_X \otimes \det(F^\vee)$ on X .

1.2.3. Given a vector bundle $p : F \rightarrow X$, every class $c \in H^1(X, F)$ coincides via the canonical isomorphism $H^1(X, F) \simeq \text{Ext}^1(\mathbb{A}_X^1, F)$ with the isomorphism class of an extension $0 \rightarrow F \rightarrow E \rightarrow \mathbb{A}_X^1 \rightarrow 0$ of vector bundles on X , where $\mathbb{A}_X^1 = X \times \mathbb{A}^1$ denotes the trivial line bundle on X . The inclusion $F \hookrightarrow E$ induces a closed immersion of $\mathbb{P}(F)$ into $\overline{\nu} : \mathbb{P}(E) \rightarrow X$ as the zero locus of the regular section of $\mathcal{O}_{\mathbb{P}(E)}(1)$ deduced from the composition $\mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow \overline{\nu}^* E \rightarrow \overline{\nu}^* \mathbb{A}_X^1$, and the complement $\mathbb{P}(E) \setminus \mathbb{P}(F)$ is then isomorphic as a scheme over X to the total space of an F -torsor $\nu : V \rightarrow X$ with isomorphism class c . In particular, the trivial extension $E = F \oplus \mathbb{A}_X^1$ corresponds to the canonical open immersion of F into $\mathbb{P}(F \oplus \mathbb{A}_X^1)$. More generally, given a line bundle $L \rightarrow X$ on X and a short exact sequence of vector bundles $0 \rightarrow F \rightarrow E \rightarrow L \rightarrow 0$, the complement of $\mathbb{P}(F) \simeq \mathbb{P}(F \otimes L^{-1})$ in $\mathbb{P}(E) \simeq \mathbb{P}(E \otimes L^{-1})$ inherits the structure of an $F \otimes L^{-1}$ -torsor with isomorphism class in $H^1(X, F \otimes L^{-1}) \simeq \text{Ext}^1(L, F)$ given by the class of the extension $0 \rightarrow F \rightarrow E \rightarrow L \rightarrow 0$.

1.3. Affine-linear bundles.

1.3.1. Affine-linear bundles over a scheme X from a sub-class of the class of locally trivial \mathbb{A}^n -bundle over X , namely, an *affine-linear bundle* of rank $r \geq 1$ over X is an X -scheme $\nu : V \rightarrow X$ for which there exists a open covering $\{U_i\}_{i \in I}$ of X and a collection of isomorphisms $\tau_i : V|_{U_i} \xrightarrow{\sim} \mathbb{A}_{U_i}^r$ of schemes over U_i such that for every $i, j \in I$, $\tau_{ij} = \tau_i \circ \tau_j^{-1}|_{U_i \cap U_j}$ is an affine automorphism of $\mathbb{A}_{U_i \cap U_j}^r = \text{Spec}_{U_i \cap U_j}(\mathcal{O}_{U_i \cap U_j}[x_1, \dots, x_r])$. This means that there exists $(A_{ij}, T_{ij}) \in \text{Aff}_r(U_i \cap U_j) = \text{GL}_r(U_i \cap U_j) \rtimes \mathbb{G}_a^r(U_i \cap U_j)$ such that $\tau_{ij}(x_1, \dots, x_r) = A_{ij}(x_1, \dots, x_r) + T_{ij}$ for every $i, j \in I$. It follows that isomorphism classes of affine-linear bundles of rank r are in one-to-one correspondence with that of principal homogeneous bundles under the affine group $\text{Aff}_r = \text{GL}_r \rtimes \mathbb{G}_a^r$.

1.3.2. Of course, every torsor under a vector bundle of rank $r \geq 1$ is an affine linear bundle of rank r . Conversely, let $\nu : V \rightarrow X$ be an affine-linear bundle of rank $r \geq 1$ with trivializations $\tau_i : V|_{U_i} \xrightarrow{\sim} \mathbb{A}_{U_i}^r$. Then for every triple of indices $i, j, k \in I$, the identities

$$\begin{cases} A_{ik} &= A_{jk} A_{ij} \\ {}^t T_{ik} &= A_{jk} \cdot {}^t T_{ij} + {}^t T_{jk} \end{cases}$$

hold in $\text{GL}_r(U_i \cap U_j \cap U_k)$ and $\mathbb{G}_a^r(U_i \cap U_j \cap U_k)$ respectively. If we identify $\text{Aff}_r(U_i \cap U_j)$ with the sub-group of

$\text{GL}_{r+1}(U_i \cap U_j)$ consisting of matrices of the form $\tilde{A}_{ij} = \begin{pmatrix} A_{ij} & {}^t T_{ij} \\ 0 & 1 \end{pmatrix}$ these relations say equivalently that $(\tilde{A}_{ij})_{i,j \in I}$

and $(A_{ij})_{i,j \in I}$ are Čech cocycles with value in GL_{r+1} and GL_r respectively for the open covering $\{U_i\}_{i \in I}$ of X . These define respectively a vector bundle $E \rightarrow X$ of rank $r+1$ and a sub-vector bundle F of E fitting in a short exact sequence $0 \rightarrow F \rightarrow E \rightarrow \mathbb{A}_X^1 \rightarrow 0$ of vector bundles on X . By construction, $\nu : V \rightarrow X$ is isomorphic as a scheme over X to the complement of $\mathbb{P}(F)$ in $\overline{\nu} : \mathbb{P}(E) \rightarrow X$, whence can be equipped with the structure of an F -torsor. Changing the trivializations τ_i by means of affine automorphisms changes the cocycle $(A_{ij})_{i,j \in I}$ for a cohomologous one and the cohomology class in $H^1(X, \text{GL}_r)$ is unaltered if we replace $\nu : V \rightarrow X$ by an isomorphic affine-linear bundle. Therefore, the vector bundle F for which an affine-linear bundle can be equipped with the structure of an F -torsor is uniquely determined up to isomorphism. Similarly, the class in $\text{Ext}^1(\mathbb{A}_X^1, F)$ defined by $(\tilde{A}_{ij})_{i,j \in I}$ is unaltered if we change the τ_i 's or replace $\nu : V \rightarrow X$ by an isomorphic affine-linear bundle, and so the isomorphism class in $H^1(X, F) \simeq \text{Ext}^1(\mathbb{A}_X^1, F)$ of $\nu : V \rightarrow X$ as an F -torsor is also uniquely determined.

2. AFFINE-LINEAR BUNDLES OVER THE AFFINE WITH A DOUBLE ORIGIN

The affine line with a double origin is the scheme $\delta : X \rightarrow \mathbb{A}^1 = \text{Spec}(\mathbb{C}[x])$ obtained by gluing two copies X_\pm of the affine line $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[x])$, with respective origins o_\pm , by the identity along the open subsets $X_\pm^* = X_\pm \setminus \{o_\pm\}$. It comes equipped with a canonical covering \mathcal{U} by the open subsets X_+ and X_- . The morphism δ is induced by the identity morphism on X_\pm and restricts to an isomorphism $X \setminus \{o_\pm\} \simeq \text{Spec}(\mathbb{C}[x^{\pm 1}])$.

Since every automorphism of X is induced by an automorphism of $X_+ \sqcup X_-$ of the form $X_\pm \ni x \mapsto ax \in X_{\varepsilon, \pm}$, where $a \in \mathbb{C}^*$ and $\varepsilon = \pm 1$, the automorphism group $\text{Aut}(X)$ of X is isomorphic to $\mathbb{G}_m \times \mathbb{Z}_2$. In what follows, we denote by $\theta = (1, -1) \in \mathbb{G}_m \times \mathbb{Z}_2$ the automorphism which exchanges the open subsets X_\pm of X .

2.1. Vector bundles on the affine line with a double origin.

2.1.1. Since every line bundle on X becomes trivial on the canonical covering \mathcal{U} , the Picard group $\text{Pic}(X)$ of X is isomorphic to $\check{H}^1(\mathcal{U}, \mathcal{O}_X^*) \simeq \mathbb{C}[x^{\pm 1}]^* / \mathbb{C}^* \simeq \mathbb{Z}$. In what follows we fix as a generator for $\text{Pic}(X)$ the class of the line bundle $p : \mathbb{L} \rightarrow X$ with trivializations $\mathbb{L}|_{X_\pm} \simeq \text{Spec}(\mathbb{C}[x][u_\pm])$ and transition isomorphism $\tau_\pm : X_+^* \times \mathbb{A}^1 \xrightarrow{\sim} X_-^* \times \mathbb{A}^1$, $(x, u_+) \mapsto (x, xu_+)$. The pull-back of \mathbb{L} by the automorphism θ of X which exchanges the two open subsets X_\pm of X is isomorphic to the dual \mathbb{L}^{-1} of \mathbb{L} . A line bundle $L \rightarrow X$ isomorphic to \mathbb{L}^k for some $k \in \mathbb{Z}$ is said to be of *degree* $-k$.

More generally, every vector bundle $E \rightarrow X$ of rank $r \geq 2$ becomes trivial on the canonical covering \mathcal{U} of X , whence is determined up to isomorphism by the equivalence class of a matrix $M \in \text{GL}_r(\mathbb{C}[x^{\pm 1}])$ in the double quotient $\check{H}^1(\mathcal{U}, \text{GL}_r) \simeq \text{GL}_r(\mathbb{C}[x]) \backslash \text{GL}_r(\mathbb{C}[x^{\pm 1}]) / \text{GL}_r(\mathbb{C}[x])$. Since for a suitable $n \geq 0$, $E \otimes \mathbb{L}^n$ is determined by a matrix $M \in \mathcal{M}_r(\mathbb{C}[x]) \cap \text{GL}_r(\mathbb{C}[x^{\pm 1}])$ equivalent in the double quotient $\text{GL}_r(\mathbb{C}[x]) \backslash \mathcal{M}_r(\mathbb{C}[x]) / \text{GL}_r(\mathbb{C}[x])$ to its Smith diagonal normal form, it follows that E splits into a direct sum of line bundles, i.e., is isomorphic to $\bigoplus_{i=1}^r \mathbb{L}^{k_i}$ for suitable $k_1, \dots, k_r \in \mathbb{Z}$. The Grothendieck group $K_0(X)$ of vector bundles on X is described as follows:

Lemma 2.1. *The map $K_0(X) \rightarrow \text{Pic}(X) \oplus \mathbb{Z}$, $[E] \mapsto (\det E, \text{rk}(E))$ is an isomorphism of groups.*

Proof. The only non trivial part is to show that this map is injective, or equivalently, that if $E \rightarrow X$ is a vector bundle of rank $r \geq 2$ then E and $\det(E) \oplus \mathbb{A}_X^{r-1}$ have the same class in $K_0(X)$. We proceed by induction on $r \geq 2$. Since every vector bundle on X is decomposable, we may assume that $E = \mathbb{L}^m \oplus E'$ where $k \in \mathbb{Z}$ and E' is a vector bundle of rank $r - 1$. By induction hypothesis, E' has the same class in $K_0(X)$ as $\det(E') = \mathbb{L}^n \oplus \mathbb{A}_X^{r-2}$ where $n = \deg(E')$, and so it is enough to check that for every $m, n \in \mathbb{Z}$, $\mathbb{L}^m \oplus \mathbb{L}^n$ and $\mathbb{L}^{m+n} \oplus \mathbb{A}_X^1$ have the same class in $K_0(X)$. If either m or n is equal to zero then we are done. Otherwise, up to changing $\mathbb{L}^m \oplus \mathbb{L}^n$ for its dual and exchanging the roles of m and n , we may assume that either $0 < m \leq n$ or $m < 0 < n$. In the first case, the matrix

$$M = \begin{pmatrix} x^m & 1 \\ 0 & x^n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & x^{n-1} \end{pmatrix} \begin{pmatrix} x^{m+n} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{m-1} & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[x^{\pm 1}])$$

is equivalent in $\check{H}^1(\mathcal{U}, \mathrm{GL}_2)$ to $\mathrm{diag}(x^{m+n}, 1)$ and defines an extension $0 \rightarrow \mathbb{L}^m \rightarrow \mathbb{L}^{m+n} \oplus \mathbb{A}_X^1 \rightarrow \mathbb{L}^n \rightarrow 0$. Hence $\mathbb{L}^m \oplus \mathbb{L}^n$ and $\mathbb{L}^{m+n} \oplus \mathbb{A}_X^1$ have the same class in $K_0(X)$. The second case follows from the same argument using the fact that the matrix

$$N = \begin{pmatrix} 1 & x^m \\ 0 & x^{m+n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^n & 1 \end{pmatrix} \begin{pmatrix} x^m & 0 \\ 0 & x^n \end{pmatrix} \begin{pmatrix} x^{-m} & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[x^{\pm 1}])$$

is equivalent in $\check{H}^1(\mathcal{U}, \mathrm{GL}_2)$ to $\mathrm{diag}(x^m, x^n)$ and defines an extension $0 \rightarrow \mathbb{A}_X^1 \rightarrow \mathbb{L}^m \oplus \mathbb{L}^n \rightarrow \mathbb{L}^{m+n} \rightarrow 0$. \square

2.2. Affine-linear bundles of rank one.

Here we review the classification of affine-linear bundle of rank one over X following [1, 3].

2.2.1. In view of the above description of $\mathrm{Pic}(X)$, every affine-linear bundle $\rho : S \rightarrow X$ of rank one over X is an \mathbb{L}^k -torsor for a certain $k \in \mathbb{Z}$. We deduce from the isomorphism $H^1(X, \mathbb{L}^k) \simeq \check{H}^1(\mathcal{U}, \mathbb{L}^k) \simeq \mathbb{C}[x^{\pm 1}] / \langle x^k \mathbb{C}[x] + \mathbb{C}[x] \rangle$ that every nontrivial \mathbb{L}^k -torsor $\rho : S \rightarrow X$ is isomorphic to a one obtain by gluing $X_+ \times \mathbb{A}^1$ and $X_- \times \mathbb{A}^1$ over $X_+ \cap X_-$ by an isomorphism of the form $(x, u_+) \mapsto (x, x^k u_+ + g(x))$ for a Laurent polynomial $g(x) \in \mathbb{C}[x^{\pm 1}]$ with non zero residue class in $\mathbb{C}[x^{\pm 1}] / \langle x^k \mathbb{C}[x] + \mathbb{C}[x] \rangle$. This implies in turn that the total space of a nontrivial \mathbb{L}^k -torsor is an affine surface. Indeed, writing $g = x^{-l} h(x)$ where $h \in \mathbb{C}[x] \setminus x\mathbb{C}[x]$ and $l > \min(0, -k)$, the local regular functions

$$\varphi_+ = x^{k+l} u_+ + h(x) \in \Gamma(S|_{X_+}, \mathcal{O}_S) \quad \text{and} \quad \varphi_- = x^l u_- \in \Gamma(S|_{X_-}, \mathcal{O}_S)$$

glue to a global one $\varphi \in \Gamma(S, \mathcal{O}_S)$ for which the morphism $\pi = (\delta \circ \rho, \varphi) : S \rightarrow \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x, y])$ maps the fibers $\rho^{-1}(o_{\pm})$ to the distinct points $(0, h(0))$ and $(0, 0)$ respectively and restricts to an isomorphism $S \setminus \rho^{-1}(\{o_{\pm}\}) \simeq \mathrm{Spec}(\mathbb{C}[x^{\pm 1}, y])$. Since the inverse images by π of the principal affine open subsets $y \neq h(0)$ and $y \neq 0$ of \mathbb{A}^2 are principal open subsets of $S \setminus \rho^{-1}(o_+) \simeq \mathbb{A}^2$ and $S \setminus \rho^{-1}(o_-) \simeq \mathbb{A}^2$ respectively, it follows that $\pi : S \rightarrow \mathbb{A}^2$ is an affine morphism whence that S is an affine scheme. Note that conversely the total space of a trivial \mathbb{L}^k -torsor cannot be affine since it is not even separated.

Example 2.2. For every $d \in \mathbb{Z}$, we let $\zeta_d : S(d) \rightarrow X$ be the nontrivial \mathbb{L}^d -torsor with gluing isomorphism

$$X_+ \times \mathbb{A}^1 \supset X_+^* \times \mathbb{A}^1 \xrightarrow{\sim} X_-^* \times \mathbb{A}^1 \subset X_- \times \mathbb{A}^1, (x, u_+) \mapsto (x, x^d u_+ + x^{\min(-1, d-1)}).$$

One checks easily that $\zeta_{-d} : S(-d) \rightarrow X$ is isomorphic to the pull-back $S(d) \times_X X$ of $\zeta_d : S(d) \rightarrow X$ by the automorphism θ of X which exchanges the open subsets X_{\pm} . Since Ω_X^1 is trivial it follows that $K_{S(d)} \simeq \Omega_{S(d)/X}^1 \simeq \zeta_d^* \mathbb{L}^{-d}$ whence that $\mathrm{Pic}(S(d)) / \langle K_{S(d)} \rangle \simeq \mathbb{Z} / |d| \mathbb{Z}$. Therefore, the surfaces $S(d)$ are pairwise non isomorphic as schemes over X while $S(d)$ and $S(d')$ are isomorphic as abstract schemes if and only if $d = \pm d'$. Note that since $\mathrm{Pic}(S(d)) \simeq \mathrm{Pic}(S(d) \times_X \mathbb{A}_X^r)$ for every $r \geq 1$, the same argument shows more generally that $S(d) \times_X \mathbb{A}_X^r$ is isomorphic to $S(d') \times_X \mathbb{A}_X^r$ as a scheme over X if and only if $d = d'$, and as an abstract scheme if and only if $d = \pm d'$.

If $d \geq 0$ then, letting $\varphi \in \Gamma(S(d), \mathcal{O}_{S(d)})$ be defined locally by $(\varphi_+, \varphi_-) = (x^{d+1} u_+ + 1, x u_-)$ as in 2.2.1, one checks that the rational functions $\psi = x^{-1} \varphi(\varphi - 1)$ and $\xi = x^{-d} \varphi^d \psi$ on $S(d)$ are regular and that the morphism $(\delta \circ \zeta_d, \varphi, \psi, \xi) : S(d) \rightarrow \mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, u])$ is a closed embedding of $S(d)$ as the surface defined by the equations

$$xz = y(y - 1), (y - 1)^d u = z^{d+1}, x^d u = y^d z.$$

The following result shows that for a non trivial affine-linear bundle of rank one $\rho : S \rightarrow X$, the isomorphism type of S as an abstract scheme is essentially uniquely determined by its one as an affine-linear bundle over X :

Theorem 2.3. *Two non trivial affine-linear bundles of rank one $\rho_i : S_i \rightarrow X$, $i = 1, 2$, have isomorphic total spaces if and only if their isomorphism classes in $H^1(X, \mathrm{Aff}_1)$ belong to the same orbit of the action of $\mathrm{Aut}(X)$.*

Proof. The condition is clearly sufficient. Conversely, if either S_1 or S_2 admits a unique affine-linear bundle structure over X up to composition by automorphisms of X then both admit a unique such structure and so every isomorphism $\Phi : S_2 \xrightarrow{\sim} S_1$ descends to an automorphism φ of X such that $\rho_1 \circ \Phi = \varphi \circ \rho_2$. This implies in turn that Φ factors through an isomorphism of affine-linear bundles $\Phi' : S_2 \rightarrow S_1 \times_X X$ whence that the isomorphism classes in $H^1(X, \mathrm{Aff}_1)$ of the Aff_1 -bundles associated to S_1 and S_2 belong to a same orbit of $\mathrm{Aut}(X)$. Otherwise, if S_1 and S_2 both admit at least two affine-linear bundle structures over X with distinct general fibers then, by combining Theorem 3.11 and 5.3 in [1], we obtain the following : if the canonical bundles K_{S_1} and K_{S_2} are both trivial then $\rho_i : S_i \rightarrow X$, $i = 1, 2$, are both isomorphic to $\zeta_0 : S(0) \rightarrow X$. Otherwise, for $d = \mathrm{ord}(\mathrm{Pic}(S_1) / \langle K_{S_1} \rangle) = \mathrm{ord}(\mathrm{Pic}(S_2) / \langle K_{S_2} \rangle)$, $\rho_i : S_i \rightarrow X$ is isomorphic as an affine-linear bundle either to the one $\zeta_d : S(d) \rightarrow X$ or to the one $\zeta_{-d} : S(-d) \rightarrow X$. This completes the proof since the latter are obtained from each others via the base change by the automorphism θ of X (see Example 2.2 above). \square

2.3. Affine-linear bundles of higher ranks.

In this subsection, we consider affine-linear bundles $\rho : V \rightarrow X$ of rank $r \geq 2$. Recall that to every such bundle is associated a vector bundle $E \rightarrow X$ unique up to isomorphism for which $\rho : V \rightarrow X$ inherits the structure of an E -torsor. In contrast with the case of affine-linear bundles of rank one, we have the following characterization:

Theorem 2.4. *Let $p_i : E_i \rightarrow X$, $i = 1, 2$ be vector bundles of the same rank $r \geq 2$ and let $\rho_i : V_i \rightarrow X$ be non trivial E_i -torsors, $i = 1, 2$. Then the following holds:*

- 1) V_1 and V_2 are isomorphic as schemes over X if and only if $\deg(E_1) = \deg(E_2)$,
- 2) V_1 and V_2 are isomorphic as abstract schemes if and only if $\deg(E_1) = \pm \deg(E_2)$.

2.3.1. Theorem 2.4 is a consequence of Lemmas 2.5 and Proposition 2.6 below which, combined with Lemma 2.1, imply that the total space of non trivial E -torsor $\rho : V \rightarrow X$ of rank $r \geq 2$ is isomorphic as a scheme over X to $S(d) \times_X \mathbb{A}_X^{r-1}$, where $d = -\deg(E)$ and where $\zeta_d : S(d) \rightarrow X$ is the non trivial \mathbb{L}^d -torsor defined in Example 2.2 above.

Lemma 2.5. *The total spaces of all non trivial torsors under a fixed vector bundle $p : E \rightarrow X$ of rank $r \geq 2$ are affine and isomorphic as schemes over X .*

Proof. By virtue of 2.1.1 above, we may assume that $E = \bigoplus_{i=1}^r \mathbb{L}^{k_i}$, $k_1, \dots, k_r \in \mathbb{Z}$. Given a non trivial E -torsor $\rho : V \rightarrow X$ there exists an index i such that the i -th component of the isomorphism class (v_1, \dots, v_r) of V in $H^1(X, E) \simeq \bigoplus_{i=1}^r H^1(X, \mathbb{L}^{k_i})$ is not zero. Up to a permutation, we may assume from now on that $v_1 \neq 0$. The actions of \mathbb{L}^{k_1} and $E_1 = \bigoplus_{i=2}^r \mathbb{L}^{k_i} \simeq E/\mathbb{L}^{k_1}$ on V commute and we have a cartesian square

$$\begin{array}{ccc} V & \xrightarrow{\pi_1} & S_1 = V/E_1 \\ \downarrow & & \downarrow \rho_1 \\ V/\mathbb{L}^{k_1} & \xrightarrow{\quad} & X \end{array}$$

where $\rho_1 : S_1 \rightarrow X$ is an \mathbb{L}^{k_1} -torsor with isomorphism class $v_1 \in H^1(X, \mathbb{L}^{k_1})$ and where $\pi_1 : V \rightarrow S_1 = V/E_1$ is a $\rho_1^* E_1$ -torsor. Since $\rho_1 : S_1 \rightarrow X$ is a non trivial torsor, S_1 is an affine scheme by virtue of 2.2.1 and so $\pi_1 : V \rightarrow S_1$ is isomorphic as a scheme over X to the total space of the trivial $\rho_1^* E_1$ -torsor $p_1 : S_1 \times_X E_1 \rightarrow S_1$. In particular, V is an affine scheme.

With the notation of Example 2.2 above, we claim that $S_1 \times_X E_1$ is isomorphic as a scheme over X to the r -fold fiber product $S(k_1) \times_X \dots \times_X S(k_r)$. Indeed, since for every $k \in \mathbb{Z}$, $\zeta_k : S(k) \rightarrow X$ is an \mathbb{L}^k -torsor, the fiber product $S_1 \times_X S(k)$ is simultaneously the total space of a $\rho_1^* \mathbb{L}^k$ -torsor over S_1 and of a $\zeta_k^* \mathbb{L}^{k_1}$ -torsor over $S(k)$ via the first and the second projection respectively. The fact that S_1 and $S(k)$ are both affine implies that the latter are both trivial torsors, which yields isomorphisms $S_1 \times_X \mathbb{L}^k \simeq S_1 \times_X S(k) \simeq \mathbb{L}^{k_1} \times_X S(k)$ of schemes over X . The same argument applied to the non trivial \mathbb{L}^{k_1} -torsor $\zeta_{k_1} : S(k_1) \rightarrow X$ provides isomorphisms $S(k_1) \times_X \mathbb{L}^k \simeq S(k_1) \times_X S(k) \simeq \mathbb{L}^{k_1} \times_X S(k)$ of schemes over X . Letting $E_2 = E_1/\mathbb{L}^{k_2} \simeq \bigoplus_{i=3}^r \mathbb{L}^{k_i}$, we finally obtain isomorphisms

$$S_1 \times_X E_1 \simeq S(k_1) \times_X S(k_2) \times_X E_2 \simeq S(k_1) \times_X S(k_2) \times_X (S(k_3) \times_X \dots \times_X S(k_r))$$

where the last isomorphism follows from the fact that the fiber product of the affine scheme $q : S(k_1) \times_X S(k_2) \rightarrow X$ with the E_2 -torsor $S(k_3) \times_X \dots \times_X S(k_r)$ is isomorphic to the trivial $q^* E_2$ -torsor $S(k_1) \times_X S(k_2) \times_X E_2$ over $S(k_1) \times_X S(k_2)$. \square

Proposition 2.6. *The isomorphism type of the total space of a non trivial affine-linear bundle $\rho : V \rightarrow X$ of rank $r \geq 2$ as a scheme over X depends only on the class in $K_0(X)$ of its associated vector bundle.*

Proof. Given a vector bundle $E \rightarrow X$ of rank $r \geq 2$, we will show more precisely that the total space of a non trivial E -torsor $\rho : V \rightarrow X$ is isomorphic as a scheme over X to $S(d) \times_X \mathbb{A}_X^{r-1}$, where $d = -\deg(E)$. We proceed by induction on the rank of E . By combining 2.1.1 and Lemma 2.5 above, we may assume that $E = \bigoplus_{i=1}^r \mathbb{L}^{k_i}$, where $k_1, \dots, k_r \in \mathbb{Z}$ and $-k_1 - \dots - k_r = -d$ and that $V = S(k_1) \times_X \dots \times_X S(k_r)$. Furthermore, since the induction hypothesis implies that $S(k_2) \times_X \dots \times_X S(k_r) \simeq S(d - k_1) \times_X \mathbb{A}_X^{r-1}$ as schemes over X , it is enough to show that for every $m, n \in \mathbb{Z}$, $S(m) \times_X S(n)$ and $S(m+n) \times_X \mathbb{A}_X^1$ are isomorphic as schemes over X . If m or n is equal to zero then we are done. Otherwise, up to taking the pull-back of $S(m) \times_X S(n)$ by the automorphism θ of X and exchanging the roles of m and n , we may assume similarly as in the proof of Lemma 2.1 above that either $0 < m \leq n$ or $m < 0 < n$. In the first case, letting $E \simeq \mathbb{L}^{m+n} \oplus \mathbb{A}_X^1$ be the vector bundle on X defined by the matrix

$$M = \begin{pmatrix} x^m & 1 \\ 0 & x^n \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[x^{\pm 1}]),$$

it follows from Lemma 2.5 that the total space of the non trivial E -torsor $\rho : V \rightarrow X$ with gluing isomorphism

$$\begin{aligned} X_+ \times \mathbb{A}^2 \supset X_+^* \times \mathbb{A}^2 &\xrightarrow{\sim} X_-^* \times \mathbb{A}^2 \subset X_- \times \mathbb{A}^2, \\ (x, (v_+, u_+)) &\mapsto (x, (x^m v_+ + u_+, x^n u_+ + x^{-1})) \end{aligned}$$

is isomorphic to $S(m+n) \times_X \mathbb{A}_X^1$. On the other hand, since E is an extension of \mathbb{L}^n by \mathbb{L}^m , V inherits a free action of \mathbb{L}^m whose quotient V/\mathbb{L}^m coincides with the total space of the \mathbb{L}^n -torsor $\zeta_n : S(n) \rightarrow X$ with gluing isomorphism $(x, u_+) \mapsto (x, x^n u_+ + x^{-1})$. Furthermore, the quotient morphism $V \rightarrow V/\mathbb{L}^m \simeq S(n)$ inherits the structure of a $\zeta_n^* \mathbb{L}^m$ -torsor whence is isomorphic to the trivial one $S(n) \times_X \mathbb{L}^m$ as $S(n)$ is affine. Summing up, we obtain isomorphisms $S(m+n) \times_X \mathbb{A}_X^1 \simeq V \simeq S(n) \times_X \mathbb{L}^m \simeq S(n) \times_X S(m)$ of schemes over X .

The case $m < 0 < n$ follows from a similar argument starting from the vector bundle $E \simeq \mathbb{L}^m \oplus \mathbb{L}^n$ defined by the matrix

$$N = \begin{pmatrix} 1 & x^m \\ 0 & x^{m+n} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[x^{\pm 1}]),$$

and the non trivial E -torsor $\rho : V \rightarrow X$ with gluing isomorphism

$$\begin{aligned} X_+ \times \mathbb{A}^2 \supset X_+^* \times \mathbb{A}^2 &\xrightarrow{\sim} X_-^* \times \mathbb{A}^2 \subset X_- \times \mathbb{A}^2 \\ (x, (v_+, u_+)) &\mapsto (x, (v_+ + x^m u_+, x^{m+n} u_+ + x^{\min(-1, m+n-1)})). \end{aligned}$$

□

3. ISOMORPHY TYPES OF COMPLEMENTS OF HYPERPLANE SUB-BUNDLES

In this section, we consider total spaces of non trivial affine-linear bundles $\nu : V \rightarrow \mathbb{P}^1$ over the projective line. We first review the case of affine-linear bundles of rank one: the crucial observation there is the fact that the total space of a non trivial $\mathcal{O}_{\mathbb{P}^1}(-d)$ -torsor $\nu : V \rightarrow \mathbb{P}^1$, where $d \geq 2$, is isomorphic to the affine surface $\zeta_{d-2} : S(d-2) \rightarrow X$ of Example 2.2, whence admits the structure of a non trivial \mathbb{L}^{d-2} -torsor over the affine line with a double origin. This enables to consider total spaces of non trivial affine-linear bundles $\nu : V \rightarrow \mathbb{P}^1$ of higher ranks as being simultaneously that of certain non trivial affine-linear bundles over X , and to deduce the classification of total spaces of such bundles as a particular case of the results established in the first section.

3.1. Affine linear bundles of rank one and the Danilov-Gizatullin Theorem.

The Danilov-Gizatullin Theorem [5, Theorem 5.8.1] asserts that the isomorphism type of the complement of an ample section C in a Hirzebruch surface $\pi_n : \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \rightarrow \mathbb{P}^1$, $n \geq 0$, depends only on the self-intersection $(C^2) \geq 2$ of C . Since there is a one-to-one correspondence between non trivial $\mathcal{O}_{\mathbb{P}^1}(-d)$ -torsors $\nu : V \rightarrow \mathbb{P}^1$ and complements of ample sections C with self-intersection $(C^2) = d$ in Hirzebruch surfaces [5, Remark 4.8.6], the Danilov-Gizatullin Theorem can be rephrased as the fact that the isomorphism type of the total space of a non trivial $\mathcal{O}_{\mathbb{P}^1}(-d)$ -torsor $\nu : V \rightarrow \mathbb{P}^1$ depends only on d and not on its isomorphism class as a torsor in $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$. We have the following more effective result:

Proposition 3.1. *The total space of a non trivial $\mathcal{O}_{\mathbb{P}^1}(-d)$ -torsor $\nu : V \rightarrow \mathbb{P}^1$, $d \geq 2$, is isomorphic to the surface $\zeta_{d-2} : S(d-2) \rightarrow X$ of Example 2.2. Furthermore the isomorphism $V \simeq S(d-2)$ can be chosen in such a way that $\nu^* \mathcal{O}_{\mathbb{P}^1}(1) = \zeta_{d-2}^* \mathbb{L}^{-1}$ in $\mathrm{Pic}(V) \simeq \mathbb{Z}$.*

Proof. Recall that for every $d \geq 2$, $\zeta_{d-2} : S(d-2) \rightarrow X$ is isomorphic to the surface in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, u])$ defined by the equations

$$xz = y(y-1), (y-1)^{d-2}u = z^{d-1}, x^{d-2}u = y^{d-2}z.$$

Letting $\mathbb{P}^1 = \mathrm{Proj}(\mathbb{C}[w_0, w_1])$, $U_0 = \mathbb{P}^1 \setminus \{[1 : 0]\} = \mathrm{Spec}(\mathbb{C}[w])$ and $U_\infty = \mathbb{P}^1 \setminus \{[0 : 1]\} = \mathrm{Spec}(\mathbb{C}[w'])$ where $w = w_0/w_1$ and $w' = w_1/w_0$, one checks that the morphism $\nu_{d-2} : S(d-2) \rightarrow \mathbb{P}^1$, $(x, y, z, u) \mapsto [x : y] = [y-1 : z]$ defines an $\mathcal{O}_{\mathbb{P}^1}(-d)$ -torsor with local trivializations

$$\tau_0 : \nu_{d-2}^{-1}(U_0) \xrightarrow{\sim} \mathrm{Spec}(\mathbb{C}[w][u]), \quad \tau_\infty : \nu_{d-2}^{-1}(U_\infty) \xrightarrow{\sim} \mathrm{Spec}(\mathbb{C}[w'][x])$$

and transition isomorphism $\tau_\infty \circ \tau_0^{-1} |_{U_0 \cap U_\infty}$ given by $(w, u) \mapsto (w', x) = (w^{-1}, w^d u + w)$. Furthermore, it follows from the construction of $\zeta_{d-2} : S(d-2) \rightarrow X$ and $\nu_{d-2} : S(d-2) \rightarrow \mathbb{P}^1$ that $\nu_{d-2}^{-1}([0 : 1]) = \zeta_{d-2}^{-1}(o_+)$. Since the classes of these divisors in $\mathrm{Cl}(S(d-2)) \simeq \mathrm{Pic}(S(d-2))$ coincide respectively with the line bundles $\nu_{d-2}^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $\zeta_{d-2}^* \mathbb{L}^{-1}$, the assertion follows from the “refined” Danilov-Gizatullin Theorem [2, Theorem 3.1] which asserts that if $\nu_i : V_i \rightarrow \mathbb{P}^1$, $i = 1, 2$, are non trivial $\mathcal{O}_{\mathbb{P}^1}(-d)$ -torsors, then there exists an isomorphism $f : V_1 \xrightarrow{\sim} V_2$ such that $f^*(\nu_2^* \mathcal{O}_{\mathbb{P}^1}(1)) \simeq \nu_1^* \mathcal{O}_{\mathbb{P}^1}(1)$. □

3.2. Affine-linear bundles of arbitrary ranks.

By combining our previous results, we obtain the following characterization:

Theorem 3.2. *Let $p_i : E_i \rightarrow \mathbb{P}^1$, $i = 1, 2$ be vector bundles of the same rank $r \geq 1$ and let $\nu_i : V_i \rightarrow \mathbb{P}^1$ be non trivial E_i -torsors, $i = 1, 2$. Then V_1 and V_2 are affine, and isomorphic as abstract varieties if and only if $\deg(\det E_1^\vee \otimes \Omega_{\mathbb{P}^1}^1) = \pm \deg(\det E_2^\vee \otimes \Omega_{\mathbb{P}^1}^1)$.*

Proof. The argument is very similar to the one used in the proof of Lemma 2.5 above. Recall that every vector bundle $E \rightarrow \mathbb{P}^1$ of rank $r \geq 2$ splits into a direct sum $E = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(k_i)$, $k_1, \dots, k_r \in \mathbb{Z}$, of line bundles [6]. Therefore, if $\nu : V \rightarrow \mathbb{P}^1$ is a non trivial E -torsor, then there exists an index $i \in \{1, \dots, r\}$ such that the i -th component of its isomorphism class $(a_1, \dots, a_r) \in H^1(\mathbb{P}^1, E) \simeq \bigoplus_{i=1}^r H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k_i))$ is non zero. Letting $E_i = \bigoplus_{j \neq i} \mathcal{O}_{\mathbb{P}^1}(k_j) \simeq E / \mathcal{O}_{\mathbb{P}^1}(k_i)$, the quotient of V by the induced action of E_i inherits the structure of a non trivial $\mathcal{O}_{\mathbb{P}^1}(k_i)$ -torsor $\nu_i : S_i \rightarrow \mathbb{P}^1$ with isomorphism class $a_i \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k_i))$. Furthermore, the quotient morphism $V \rightarrow S_i = V/E_i$ has the structure of a $\nu_i^* E_i$ -torsor. Proposition 3.1 above implies that $k_i = -d$ for some $d \geq 2$ and that S_i is isomorphic to the surface $\zeta_{d-2} : S(d-2) \rightarrow X$. In particular, $S_i \simeq S(d-2)$ is affine and so V is isomorphic to the trivial $\nu_i^* E_i$ -torsor $S_i \times_{\mathbb{P}^1} E_i$. Moreover, by choosing the isomorphism $S_i \simeq S(d-2)$ in such a way that $\nu_i^* \mathcal{O}_{\mathbb{P}^1}(1) \simeq \zeta_{d-2}^* \mathbb{L}^{-1}$, we obtain that $S_i \times_{\mathbb{P}^1} E_i$ is an affine variety, isomorphic as a scheme over X to the total space of a non trivial torsor under the vector bundle $\tilde{E} = \mathbb{L}^{d-2} \oplus \mathbb{L}^{-k_2} \oplus \dots \oplus \mathbb{L}^{-k_r}$. Since $\deg \tilde{E} = -\deg(\det E^\vee \otimes \Omega_{\mathbb{P}^1}^1)$, the assertion follows from Theorem 2.4 above. □

Example 3.3. Let us consider again the example given in the introduction of an affine variety which is simultaneously the total space of non trivial torsors under complex vector bundles of different topological types. The Euler exact sequence $0 \rightarrow \Omega_{\mathbb{P}^1}^1 \xrightarrow{j} \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0$ on \mathbb{P}^1 defines a non trivial $\Omega_{\mathbb{P}^1}^1$ -torsor $v : V \rightarrow \mathbb{P}^1$ with total space isomorphic to the complement of the diagonal $\Delta \simeq \mathbb{P}(\Omega_{\mathbb{P}^1}^1)$ in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. For every $k \in \mathbb{Z}$, the variety $V_k = V \times_{\mathbb{P}^1} \mathcal{O}_{\mathbb{P}^1}(k) \rightarrow \mathbb{P}^1$ is then a torsor under the vector bundle $F_k = \Omega_{\mathbb{P}^1}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(k)$. Since $\deg(\det(F_k^\vee) \otimes \Omega_{\mathbb{P}^1}^1) = k = -\deg(\det(F_k^\vee) \otimes \Omega_{\mathbb{P}^1}^1)$, Theorem 3.2 implies that V_{-k} and V_k are isomorphic affine varieties. The exact sequence

$$0 \rightarrow F_k = \Omega_{\mathbb{P}^1}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(k) \xrightarrow{j \oplus \text{id}} E_k = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(k) \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0$$

provides an open embedding of $V_k \simeq V_{-k}$ into $\mathbb{P}(E_k)$ as the complement of the hyperplane sub-bundle $H_k = \mathbb{P}(F_k)$. Note that if $k = 0$ then H_0 is nef but not ample and that if $k > 2$ then H_{-k} is ample whereas H_k has negative self-intersection $(H_k^3) = 2 - k$.

More generally, for any $n \geq 2$, the complement V in $\mathbb{P}^n \times \mathbb{P}^n = \text{Proj}(\mathbb{C}[x_0, \dots, x_n]) \times \text{Proj}(\mathbb{C}[y_0, \dots, y_n])$ of the ample divisor D with equation $\sum_{i=0}^n x_i y_i = 0$ inherits simultaneously via the first and the second projection the structure of an $\Omega_{\mathbb{P}^n}^1$ -torsor $\nu_i : V \rightarrow \mathbb{P}^n$ associated with the Euler exact sequence $0 \rightarrow \Omega_{\mathbb{P}^n}^1 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow 0$ on each factor in $\mathbb{P}^n \times \mathbb{P}^n$. Since D is of type $(1, 1)$ in $\text{Pic}(\mathbb{P}^n \times \mathbb{P}^n) \simeq \text{p}_1^* \text{Pic}(\mathbb{P}^n) \oplus \text{p}_2^* \text{Pic}(\mathbb{P}^n)$, we have for every $k \in \mathbb{Z}$, $\nu_2^* \mathcal{O}_{\mathbb{P}^n}(k) = \nu_1^* \mathcal{O}_{\mathbb{P}^n}(-k)$ in $\text{Pic}(V) \simeq \mathbb{Z}$. Therefore, similarly as in the previous case, we may interpret $V \times_{\nu_1, \mathbb{P}^n} \mathcal{O}_{\mathbb{P}^n}(-k) \simeq V \times_{\nu_2, \mathbb{P}^n} \mathcal{O}_{\mathbb{P}^n}(k)$ as being simultaneously the total space of an $\Omega_{\mathbb{P}^n}^1 \oplus \mathcal{O}_{\mathbb{P}^n}(-k)$ -torsor and of an $\Omega_{\mathbb{P}^n}^1 \oplus \mathcal{O}_{\mathbb{P}^n}(k)$ -torsor over \mathbb{P}^n via the first and the second projection respectively.

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